**ECE 7650 (Advance Matrix Algorithm)**

**REPORT**

*On*

**“FINAL EXAMINATION FALL 2016”**

*By*

**Jamiu Babatunde Mojolagbe**

(Student ID: #7804719)

The Department of Electrical and Computer Engineering

University of Manitoba

*Submitted to*

**Ian Jeffrey, PhD**

(Course Instructor)

**PROBLEM 1**

**Question 1(i)**

This question was implemented in file named “**Q1i.m**”. Random matrices of chosen dimension were used and the following results were obtained:

**Case 1:** For randomly chosen ‘**m**’

|  |  |  |
| --- | --- | --- |
| Matrix Dimension | Size of Krylov Subspace (m) | Rank of V |
| 10x10 | 5 | 6 |
| 20x20 | 12 | 13 |
| 50x50 | 35 | 12 |
| 100x100 | 98 | 8 |
| 200x200 | 131 | 7 |
| 300x300 | 211 | 7 |
| 400x400 | 313 | 1 |

**Case 2:** For fixed ‘**m**’ for different dimensions of input matrix **A**

|  |  |  |
| --- | --- | --- |
| Matrix Dimension | Size of Krylov Subspace (m) | |
| **Rank(V) for m=5** | **Rank(V) for m=10** |
| 10x10 | 6 | 10 |
| 20x20 | 6 | 11 |
| 50x50 | 6 | 11 |
| 100x100 | 6 | 11 |
| 200x200 | 6 | 11 |
| 300x300 | 6 | 11 |
| 400x400 | 6 | 11 |

**Observation:**

As it can be observed from the tables, it is obvious that choosing **V = [r Ar A2r . . . Am-1r]** resulted in linearly dependent bases **V** such that, **V** is rank deficient. While **Case 1** presented above tends to be a generic case and not totally clear, however observing **Case 2** actually led to the conclusion that for a chosen size of Krylov subspace ‘**m**’, provided that **m < dim(A)**, **rank(V) = m+1** and for **m = dim(A)**, **rank(V)** = **m** for linearly independent bases.

**Note:** *The ranks obtained are more than ‘****m****’ since dimension of* ***V*** *is actually* ***nx(m+1)*** *but in a specific case* ***Vm*** *of dimension* ***nxm*** *can be obtained from it by just removing the last column.*

**Question 1(ii)**

Arnoldi method was implemented based on Modified Gram-Schmidt process as a function called ‘**arnoldi.m**’. The driver program for this question is named ‘**Q1ii.m**’. For the sake of clarity, ‘**Raw V**’ used to refer to the bases **V** obtained from **V = [r Ar A2r . . . Am-1r]** while the ones obtained from Arnoldi iteration is termed ‘**Arnoldi V**’.

**Case 1:** For randomly chosen ‘**m**’

|  |  |  |  |
| --- | --- | --- | --- |
| Matrix Dimension | Size of Krylov Subspace (m) | Rank of | |
| **Raw V** | **Arnoldi V** |
| 10x10 | 9 | 10 | 10 |
| 20x20 | 11 | 12 | 12 |
| 50x50 | 41 | 13 | 42 |
| 100x100 | 78 | 11 | 79 |
| 200x200 | 153 | 8 | 154 |
| 300x300 | 240 | 4 | 241 |
| 400x400 | 297 | 2 | 298 |

**Case 2:** For fixed ‘**m**’ for different dimensions of input matrix **A**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Matrix Dimension | Rank | | | |
| **V for m=5** | | **V for m=10** | |
| **Raw V** | **Arnoldi V** | **Raw V** | **Arnoldi V** |
| 10x10 | 6 | 6 | 10 | 10 |
| 20x20 | 6 | 6 | 11 | 11 |
| 50x50 | 6 | 6 | 11 | 11 |
| 100x100 | 6 | 6 | 11 | 11 |
| 200x200 | 6 | 6 | 11 | 11 |
| 300x300 | 6 | 6 | 11 | 11 |
| 400x400 | 6 | 6 | 11 | 11 |

**Observation:**

From the table shown above for **Case 1**, it can be observed that using Arnoldi method, the bases **V** obtained are linearly independent such the rank of **V** are now **m+1** (for the reasons previously presented). If **Vm** is considered, that is **V** of dimension **nxm**, then **rank(V) = m**.

Hence forth, the table in **Case 2** perfectly confirm the assertion made in the “**Observation**” section of the previous question, that is **Question 1(i)** above for **Case 2**.

**Note:** *The ranks obtained are more than ‘****m****’ since dimension of* ***V*** *is actually* ***nx(m+1)*** *but in a specific case* ***Vm*** *of dimension* ***nxm*** *can be obtained from it by just removing the last column.*

**PROBLEM 2**

**Question 2(i)**

While it is necessary to say that Arnoldi process in itself does not directly access the entries of a given matrix but instead makes the matrix map vectors and as such reaches its conclusions from their images. However, it is important to show rigorously or partially rigorously that for **VmTAVm = Hm**, when **m=n** that the eigenvalues of **Hm = Hn**are equal to the eigenvalues of **A**.

Recall that square matrices say, A and B, are related by:

**B = T-1AT = TTAT**

where **T-1** => Non-singular matrix (that is, invertible)

The transformation represented by **T-1AT** above is known as similarity transformation or conjugation by matrix **T**.  
Suppose:

**A** = 0 0 **B** = 0 1

1 0 0 0

These two matrices are similar under transformation or conjugation by **T**, such that **T** is given by

**T** = 0 1

1 0

Since a linear transformation is the same as a matrix after a basis, says bi, is chosen; then it can be shown that:

**L**(Ʃλibi) **=** Ʃajiλibj

Now, changing the basis, changes the co-efficient of the matrix such that

**L**(Ʃγiei) **=** Ʃajiγiej

If operator **L**(V) = AV uses standard basis (Euclidean basis), then L is the matrix **TAT-1** with basis

**bi = Tei**

**Note**: *Notice how the eigenvalues are represented to depict that they are images of each other*.

**Question 2(ii)**

To proof that for the largest eigenvalue of **A** and **Hm** are (approximately) equal, a file named “**Q2ii.m**” was used. Random matrices were used as well as random initial vector and **Hm** was calculate from Arnoldi method previously implemented. The following results were obtained:

**Case 1:** For randomly chosen ‘**m**’

|  |  |  |  |
| --- | --- | --- | --- |
| Matrix Dimension | Size of Krylov Subspace (m) | Maximum Eigenvalue of | |
| **A** | **Hm** |
| 10x10 | 8 | -3.9721 | -3.9752 |
| 20x20 | 18 | -3.735+1.5751i | -3.7347+1.5762i |
| 50x50 | 33 | 8.3546 | 8.3544 |
| 100x100 | 93 | -2.03653+9.94841i | -2.03653+9.94841i |
| 200x200 | 171 | -2.82874+15.0757i | -2.82874+15.0757i |
| 300x300 | 266 | -4.39007+16.9614i | -4.39007+16.9614i |
| 400x400 | 333 | 19.5548+6.06034i | 19.5548+6.06034i |

**Case 2:** For fixed ‘**m**’ for different dimensions of input matrix **A**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Matrix Dimension | Maximum Eigenvalue | | | |
| **m=5** | | **m=10** | |
| **A** | **Hm** | **A** | **Hm** |
| 10x10 | 2.1836+2.2106i | -0.99642+2.7697i | -2.6912 | -2.6912 |
| 20x20 | 5.1404 | 4.9816 | 4.3142+1.3456i | -4.4952 |
| 50x50 | -8.6843 | -8.8097 | -2.5343+7.2262i | -5.2299+5.1851i |
| 100x100 | -8.62056+5.87658i | -6.8939 | 9.28855+4.45615i | 8.4777+4.5603i |
| 200x200 | 15.1315+2.55873i | -7.7372 | -8.25263+12.1035i | 8.94859+10.3865i |
| 300x300 | -9.26965+15.9366i | -10.9218+7.77496i | 18.424 | -8.13645+11.1553i |
| 400x400 | -7.43289+18.9942i | -9.83433+6.59414i | 20.7206 | -10.4521+12.1804i |

**Observation:**

Though **Case 1** uses randomly chosen ‘**m**’ but it really does so such that it is very useful for arriving at a very good conclusion here. Why? Because the randomly chosen ‘**m**’s in **Case 1** above tend to be less that the dimension of a but **are not much less**, which is good for our purpose here. Therefore, from table for **Case 1**, it can be concluded that when **m < n** but not much less, that is when the value of **m** approaches the dimension **n**, then the eigenvalues of **Hm** tend to mimic eigenvalues of **A** - and in fact some cases in **Case 1** above the eigenvalue of **A** is equal to eigenvalue of **Hm** for that given precision.

Observing **Case 2** above, however, clear things up and it can be concluded that when **m<<n** then the eigenvalues (maximums are used here) of **Hm** are not in any way approximately equals to eigenvalues of **A**.

**Question 2(iii)**

Give a matrix **Anxn**, show that when **m < n, A = VmHmV**m**T** is true or false.

Recall from Arnoldi that:

**Hm = VmTAmVm** ----------------------- (i)

where **Vm** is **nxm**

**Hm** is **mxm**

A is **nxn**

As given in the question:

**A = VmHmVmT**  ------------------------ (ii)

Put (i) into (ii)

**A =** (**VmVmT)A(VmVmT)**

When **n=m**,

then **VmVmT** = **VmTVm**= Identity matrix (**I**) => unitary matrix

therefore, **A = IAI = A**

When **n < m**

A = (**VmVmT)A(VmVmT)**

**VmV**m**T** ≠ I

thus; **A** ≠ **VmVmTAmVmVmT**

Finally, it can be concluded that for the inequality n < m, statement **A = VmHmV**m**T** is not true (that is false) for given real matrix **Anxn**.

**PROBLEM 3**

**Question 3(i)**

Deriving the directional derivative of cost functional given below in terms of the arbitrary direction vector **h** using the real scalar parameter **ɛ**:

Recall that the directional derivative is given by:

Now,

since

If **h = r**, then

**Question 3(ii)**

For step length **,** direction **d**, evaluate functional

Recall, r = b - Ax

Given that, d = r

Now, derivative of the function with respect to is equal to 0

Comparing the result obtain for above with the alpha in **Minimum Residual 1-D**, shows that both has the same formulas

**Question 3(iii)**

From 3(i) above,

Such that,

Therefore:

since **r = b – Ax**

since **d =**

The derivative of the function with respect to is equal to 0

Finally, the obtained result is identical to alpha in **Residual Norm Steepest Descent** algorithm.

**Question 4(i)**

Recall from Arnoldi that,

**Hm = VmTAmVm**

where **Vm** is **nxm**

**Hm** is **mxm**

A is **nxn** and symmetric

Let’s consider the (**ij**)th entry of matrix **Hm**

Recall that, orthonormal basis for

Thus;

Hence, = 0 for **i > j + 1**

Since,

Such that,

Since matrix **A** given is symmetric, then we have symmetric **Hm**, thus as a result of the reason presented/proven above.

Therefore, it can be concluded that for a symmetric matrix **A**, **Hm** produce from Arnoldi process is symmetric tridiagonal.